J. Fluid Mech. (1969), *vol.* 39, *part* 3, *pp.* 511–528
Printed in Great Britain

A two-layer model of Gulf Stream separation

By A. T. PARSONS†
University of Bristol

(Received 24 February 1969)

A study is made of the wind-driven circulation of a two-layer ocean within a square basin, with a view to describing the observed separation of western boundary currents. The lower layer is allowed to surface and the line along which the upper-layer depth vanishes is interpreted as the region of the surfacing thermocline. For a representative wind stress the theory predicts the gross features of the Gulf Stream flow, the region adjacent to the surfacing line containing the separated boundary current. By assuming that the effects of friction and inertia are confined to regions of a boundary-layer character, the position of a separated current is shown to depend only on the degree of stratification and certain integral properties of the applied wind stress.

1. Introduction

Since the pioneering work of Stommel (1948) it has been established that the intense western boundary currents found in the major ocean basins of the world are due to the variation of the earth's Coriolis parameter with latitude. An important feature of the real circulation that has not been adequately accounted for is the observed separation of the western boundary current from its coast of origin and its continued preservation as a concentrated stream. A well-known example of this phenomenon is afforded by the Gulf Stream in the North Atlantic Ocean. After its formation in the Gulf of Mexico and its passage through the Florida Straits, the Gulf Stream leaves the American coast at Cape Hatteras and follows roughly a north-easterly course. On reaching central ocean regions it finally breaks up into a complicated system of multiple streams known collectively as the North Atlantic Current.

The ideas that have been advanced to explain the separation phenomenon have proceeded along two more or less independent lines. The first approach (see, for example, Greenspan (1962) or Carrier & Robinson (1962)) deals with a predominantly inertial western boundary current, and the prediction of separation rests upon the basic hypothesis that this inertial layer can be joined smoothly to the interior flow. Thus in regions of outflow from the boundary current, where it is not possible to join the two regions of flow, it is deduced that the boundary current must have separated prior to entering these regions. The flaw in this argument lies in the assumption that the gross effects of even the smallest friction can be neglected. Neither in the numerical solutions of Veronis (1966), which

† Present address: Admiralty Research Laboratory, Teddington.

include frictional and inertial effects, nor in the analytical solutions of Moore (1963) is a separation of the boundary current suggested. Bryan (1963) has even investigated the effect of an abrupt discontinuity in the western coast but found that the boundary current had no inclination to permanently separate. In regions where the inertial model would predict the non-existence of a boundary current, these studies show fairly conclusively that a matching between the inertial layer and the interior flow is obtained through a complicated intermediate region in which friction is dominant. For a full discussion of the role of friction in the wind-driven ocean circulation the reader is referred to Stewart (1964, pp. 3–9).

The second approach, due to Morgan (1956), again deals with an inertial model of the boundary current, but the mechanism of separation is this time independent of the neglect of friction. Morgan considers a two-layer system with fluid of different density in each layer. He finds that it is possible for the depth of the upper layer at the western coast to vanish so that a separation of the upper layer fluid is forced. The suggestion here that the separation mechanism is that of opposing pressure gradients due to density stratification brings to mind the fact that the thermocline, usually defined by the depth of the 10 °C isotherm, does indeed surface abruptly in the real ocean on the shoreward edge of the separated current. However, since Morgan is primarily concerned with the Gulf Stream formation he does not proceed to examine the implications of his result.

The present investigation is an attempt to push the two-layer model to its logical conclusion in connexion with the separation problem. Thus the lower layer is allowed to surface and the corresponding line along which the upper layer depth vanishes is interpreted as the region of the surfacing thermocline. Then it is possible to describe in simple terms the manner in which a separated boundary current depends upon the physical parameters of the model. Throughout the analysis the important effects of friction are included.

In § 2 the basic assumptions of the model are displayed and the equations of motion set up. Some preliminary solutions are derived in § 3 for the case of a non-surfacing lower layer. These solutions are examined in § 4 to give an indication of where, and under what conditions, a surfacing lower layer is likely to occur. Section 5 deals with the full problem of surfacing. The region of the upper layer adjacent to the surfacing line is of a boundary-layer character and contains the separated boundary current. Illustrative solutions are obtained for a model of the North Atlantic circulation. These solutions are derived only for cases in which the inertial terms are negligible. However, it is shown in § 6 by general vorticity arguments that the position of a separated boundary current is insensitive to the effects of friction and inertia, being primarily a function of certain gross properties of the wind stress.

2. Basic equations

We consider the steady wind-driven ocean circulation within a square basin. Take the origin of a Cartesian co-ordinate system at the south-west corner of the basin with the x axis directed eastward and the y axis northward. The sides of the basin are at $x = 0, L$ and $y = 0, L$.

The effects of density stratification are modelled in terms of a two-layer system, the upper and lower layers of uniform density ρ_0 and ρ_1 respectively. In order to be as realistic as possible the interface is placed at about the depth of the thermocline so that the lower layer is essentially much deeper than the upper layer.

The principal assumptions that we make are as follows: (i) pressure is hydrostatic; (ii) the lower-layer fluid is inert; (iii) the effect of friction is reproduced by an interfacial drag proportional to the velocity; (iv) the horizontal components of velocity in the upper layer are uniform with depth.

Then the two-dimensional and incompressible vertically integrated equations of motion and continuity for the upper-layer fluid are written as

$$D(\mathbf{u}.\nabla)\,\mathbf{u}+\mathbf{f}\wedge D\mathbf{u} = -g'D\nabla D+\boldsymbol{\tau}/\rho_0-K\mathbf{u}, \tag{2.1}$$

$$\nabla(D\mathbf{u}) = 0, \tag{2.2}$$

where $\mathbf{u} = (u,v)$ is the horizontal velocity vector in the (x,y) frame, $\boldsymbol{\tau} = (\tau_x, \tau_y)$ is the wind-stress vector, f = Coriolis parameter of the earth, $g' = g(1-\rho_0/\rho_1)$ is an effective acceleration due to gravity, g = actual acceleration due to gravity, K = drag coefficient.

In the absence of the inertial terms the last of the above assumptions is not necessary, so long as Du and Dv in these equations are interpreted as vertically integrated components of horizontal volume transport.

A linearization of (2.1) and (2.2) produces the model studied originally by Stommel (1948). Of particular interest in the present study is the effect of the non-linear pressure term.

In accordance with the usual β plane approximation the Coriolis parameter is linearized about the central latitude of the basin, θ_0 say. Thus, if Ω is the angular velocity of the earth's rotation and R is the earth's radius,

$$f = 2\Omega[\sin\theta_0+\cos\theta_0(2y-L)/2R]. \tag{2.3}$$

To be precise, we suppose that the ocean basin is in the northern hemisphere so that f is everywhere positive in the region of interest.

It is convenient at this stage to introduce a dimensionless primed notation in terms of parameters which are characteristic of the motion. First, the independent variables are transformed by

$$x = Lx', \quad y = Ly'. \tag{2.4}$$

Then the Coriolis parameter may be written as

$$f = L\beta f', \quad f' = f_0+y', \tag{2.5}$$

where

$$\beta = 2\Omega\cos\theta_0/R, \tag{2.6}$$

$$f_0 = R\tan\theta_0/L-0{\cdot}5. \tag{2.7}$$

For the remaining variables let

$$\boldsymbol{\tau} = W\boldsymbol{\tau}', \quad D = dD', \quad \mathbf{u} = (g'd/L^2\beta)\,\mathbf{u}, \tag{2.8}$$

where W is a typical wind stress and d is the mean depth of the upper layer such that, if V is the total volume of fluid in this layer,

$$d = V/L^2. \tag{2.9}$$

The choice of scaling of the velocity vector assumes a balance between the pressure, Coriolis and wind-stress terms in the ocean interior.

In the work to follow the primes attached to dimensionless variables will be dropped. Equations (2.1) and (2.2) now become

$$R_s D(\mathbf{u}.\nabla)\,\mathbf{u}+\mathbf{f}\wedge D\mathbf{u} = -D\nabla D+\lambda\boldsymbol{\tau}-\epsilon\mathbf{u}, \tag{2.10}$$

$$\nabla.(D\mathbf{u}) = 0, \tag{2.11}$$

where the physical parameters which describe the system are incorporated in the three dimensionless numbers

$$R_s = g'd/L^4\beta^2, \quad \epsilon = K/\beta L d, \quad \lambda = LW/g'\rho_0 d^2. \tag{2.12}$$

The number ϵ is a dissipation coefficient and R_s is interpreted as a Rossby number. It is difficult to assign meaningful values to these numbers since it is not known in exactly what sense the above model approximates to the rea ocean. However, if the real ocean is governed by the dynamics of this simplified model it is necessary that both R_s and ϵ be small.

The importance of the wind stress is measured by the number λ, which is in general of order unity. It is clear that the principal role of the inert lower layer is to emphasize the effect of the wind stress through the reduced upper-layer gravity g'.

It is useful to integrate the continuity equation in terms of a stream function for the transport in the upper layer:

$$Du = -\partial\psi/\partial y, \quad Dv = \partial\psi/\partial x. \tag{2.13}$$

Then the boundary conditions to be satisfied are:

$$\psi = 0 \quad \text{at rigid boundaries,} \tag{2.14}$$

and, if the lower layer surfaces anywhere within the basin,

$$\psi = \text{constant} \quad \text{at} \quad D(x,y) = 0. \tag{2.15}$$

There is one final condition to be satisfied to complete the statement of the problem. From the depth scaling defined in (2.10) and (2.11) the dimensionless volume of the upper layer is unity. Since this value must be conserved for any choice of external parameters or forcing term one demands that

$$1 = \iint D(x,y)\,dx\,dy, \tag{2.16}$$

where the integration is taken over the total free surface area of the upper layer.

Throughout most of the work to follow we consider only zonal wind stresses of the form

$$\boldsymbol{\tau} = [\tau_x(y), 0]. \tag{2.17}$$

This simplification will not affect the principal conclusions of the study.

3. Solutions for non-surfacing lower layer

Before considering the problem of a fully developed surfacing lower layer we give a brief derivation of some solutions that are valid in the absence of any surfacing. An examination of how and where these solutions first break down will give an indication of how the position of the surfacing region depends on the wind stress.

For any given dimensionless stress function $\tau_x(y)$ one expects that when the amplitude λ, as defined in (2.12), is sufficiently large the lower layer will be forced to surface somewhere within the basin. Since the case $\lambda = 0$ corresponds to the trivial solution $D = 1$, there will always exist, for each choice of wind stress, a positive maximum value $\lambda = \lambda_c$ such that for all solutions with $0 < \lambda < \lambda_c$ the upper-layer depth nowhere vanishes. Accordingly, we suppose for the moment that λ satisfies this inequality.

If the Rossby number is sufficiently small the inertial terms in the momentum equations (2.10) may be neglected. Then we obtain

$$-f\frac{\partial\psi}{\partial x} = -D\frac{\partial D}{\partial x}+\frac{\epsilon}{D}\frac{\partial\psi}{\partial y}+\lambda\tau_x, \tag{3.1}$$

$$-f\frac{\partial\psi}{\partial y} = -D\frac{\partial D}{\partial y}-\frac{\epsilon}{D}\frac{\partial\psi}{\partial x}, \tag{3.2}$$

where the velocity components have been eliminated in favour of the stream function defined in (2.13). The boundary conditions are

$$\psi = 0 \quad \text{at} \quad x = 0, 1 \quad \text{and} \quad y = 0, 1. \tag{3.3}$$

This system of equations with the assumption of a non-surfacing lower layer is similar to the model studied by Welander (1966). Since ϵ is a small parameter an approximate solution can be obtained by the method of singular perturbations.

Except in certain boundary-layer regions the effect of the frictional terms is small and may be neglected. The only physically acceptable place for a boundary layer is at the western coast (Stewart 1964). Then the inviscid solution for the ocean interior which satisfies the condition (3.3) at the eastern coast is found to be

$$\psi = \lambda(1-x)\,d\tau_x/dy, \tag{3.4}$$

$$D^2 = h^2+2\lambda(1-x)f^2 d(\tau_x/f)/dy, \tag{3.5}$$

where h is a constant to be determined and is identified as the depth of the upper layer at the eastern coast. We make the usual assumption that $d\tau_x/dy$ is of one sign everywhere within the basin and that it is zero at the northern and southern boundaries, so that the conditions (3.3) at these boundaries are automatically satisfied.

Since the value of the interior stream function (3.4) at the western coast is in general non-zero it is necessary to include the frictional terms in our discussion. Let this value at the western coast be $\psi_1(y)$ and the corresponding depth be $D_1(y)$. A scaling analysis at the western coast suggests a boundary-layer thickness of order ϵ. The relevant boundary-layer equations are then

$$-f\frac{\partial\psi}{\partial x} = -D\frac{\partial D}{\partial x}, \tag{3.6}$$

$$-f\frac{\partial\psi}{\partial y} = -D\frac{\partial D}{\partial y} - \frac{\epsilon}{D}\frac{\partial\psi}{\partial x}, \tag{3.7}$$

with the boundary conditions

$$\psi = 0 \quad \text{at} \quad x = 0, \quad \psi \to \psi_1(y) \quad \text{as} \quad x \to \infty. \tag{3.8}$$

Integration of the geostrophic relation (3.6) gives immediately

$$\psi = (D^2 - D_0^2)/2f, \tag{3.9}$$

where $D_0(y)$ is the upper-layer depth at the western coast due to the boundary layer. Thus D_0 can be calculated from the condition of geostrophy and a knowledge of the interior solution alone, because the matching condition of (3.8) when applied to (3.9) supplies

$$D_0^2 = D_1^2 - 2f\psi_1,$$

which may be written as

$$D_0^2 = h^2 - 2\lambda\tau_x. \tag{3.10}$$

The second momentum equation is now employed to complete the boundary-layer solution. Elimination of the stream function between (3.7) and (3.9) yields the following equation for D:

$$2\epsilon\,\partial D/\partial x + D^2 = D_1^2. \tag{3.11}$$

In terms of a stretched boundary-layer co-ordinate $\eta = xD_1/\epsilon$, the required solution is

$$D = D_1(1 - B\,e^{-\eta})/(1 + B\,e^{-\eta}), \tag{3.12}$$

where $B = (D_1 - D_0)/(D_1 + D_0)$. The corresponding solution for ψ is obtained from the geostrophic relation (3.9).

Lastly, the value of h is fixed by the conservation condition (2.16). Then, for a given dimensionless wind stress, h is a function of the parameters λ and ϵ. But the error involved in neglecting contributions to this integral from the boundary layer is at most of order ϵ. Therefore, to the same accuracy that the above solutions are correct, h may be regarded as a function of λ only.

The principal feature of the above solutions is the intense crowding of the streamlines and depth contours towards the western coast. The sense of the circulation is anticlockwise for a cyclonic wind ($d\tau_x/dy < 0$), and clockwise for an anticyclonic wind ($d\tau_x/dy > 0$).

4. The minimum depth

The preceding analysis is valid provided that the upper-layer depth vanishes nowhere within the basin. It is therefore relevant to ask where does the minimum upper-layer depth occur for any particular wind stress and what is the critical value λ_c for which this minimum depth is just zero. In order to gain a feeling for the problem two special cases are considered.

Case (a): an anticyclonic wind

Suppose that the wind stress is defined by

$$\tau_x(y) = -\cos \pi y. \tag{4.1}$$

This stress exerts a negative vorticity tendency upon the ocean and approximates fairly well to the observed stress pattern over the North Atlantic Ocean, with the westerly winds over the poleward half of the basin and the easterly winds above the equator.

If this function is substituted into the interior solution (3.5) for D, one sees that a minimum depth is attained at the north-west corner of the basin, where it has the value $D_1(1)$. It remains to find whether there exists any point in the western boundary layer with depth less than this. From the boundary-layer solution (3.12) it is readily shown that D is an increasing function of x, which implies that the minimum depth inside the boundary layer is at some point of the western coast itself. Then the result (3.10) asserts that the depth D_0 at the western coast is smallest at $y = 1$, at which point it is identical with D_1. Therefore the upper layer depth attains an absolute minimum at the north-west corner of the basin, with the value

$$D_{\min} = (h^2 - 2\lambda)^{\frac{1}{2}}. \tag{4.2}$$

So the analysis of the preceding section is valid provided that $\lambda < \lambda_c$, where λ_c is the smallest root of the equation

$$h^2(\lambda) - 2\lambda = 0. \tag{4.3}$$

The dependence of h upon the dimensionless wind-stress amplitude λ has been calculated numerically from the conservation condition (2.16). Figure 1 shows graphs of both h and $D_{\min}$ as functions of λ. The minimum upper-layer depth steadily decreases with increasing amplitude until it finally vanishes at the approximate critical value $\lambda_c = 0{\cdot}26$. One would expect that if the amplitude were increased beyond this critical value a line of surfacing would advance from the north-west corner of the basin and sever the contact of the upper-layer fluid with the neighbouring coast. If this is the case, the position of the resulting surfacing line might bear some resemblance to the observed position of the surfacing thermocline in the North Atlantic Ocean. This possibility will be investigated presently.

Case (b): a cyclonic wind

In contrast with the previous example, suppose that the basin is subjected to the stress

$$\tau_x(y) = +\cos \pi y. \tag{4.4}$$

Then the wind imparts positive vorticity to the ocean.

It may be easily verified that the minimum upper-layer depth occurs on the central latitude of the basin at a point just outside the western boundary layer. Its value is

$$D_{\min} = (h^2 - 2\pi\lambda)^{\frac{1}{2}}. \tag{4.5}$$

Again, the conservation condition has been used to determine the dependence of h upon λ. It is found that h increases with λ according to an approximate linear law, forecasting a vanishing of the minimum upper-layer depth at the critical amplitude $\lambda_c = 0{\cdot}25$. This is almost identical with the critical value for the anticyclonic wind of case (a).

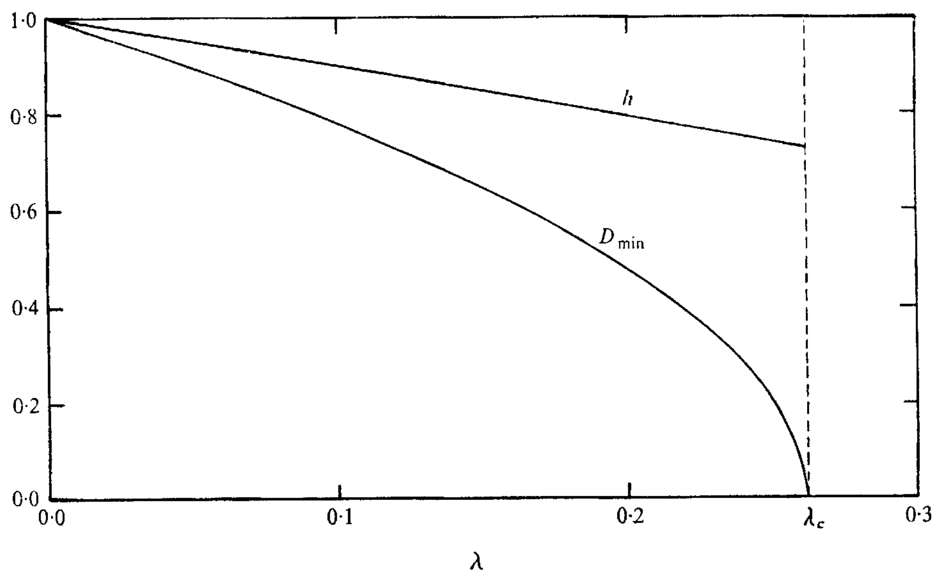

FIGURE 1. Layer depth at eastern coast and minimum layer depth as functions of amplitude. Surfacing occurs at critical amplitude λ_c.

However, a comparison of these two examples indicates a significant difference in the effects of a wind stress with negative curl and one with positive curl. In each case the critical point of surfacing is on the western side of the basin, but the position tends to be farther south in the case of a wind stress with positive curl. In the particular example cited the surfacing occurs at a latitude where the boundary current would normally be at its strongest. An interruption of the flow in this region would tend to inhibit the formation of a well-defined boundary current.

In short, these preliminary results indicate that a cyclonic wind tends to disrupt the boundary current whereas an anticyclonic wind, associated with a more northerly position of surfacing, might divert the current from the coast. Since both the Atlantic and Pacific Oceans are indeed subjected to anticyclonic stress systems, and since they both exhibit pronounced western boundary currents and separated streams, a study of anticyclonic wind stresses would seem to be more appropriate.

We now seek conditions on the stress that are sufficient to ensure a critical point of surfacing in the north-west corner of the basin. The following conditions are found to satisfy our requirements:

$$\left.\begin{array}{ll}\text{(i)} & d\tau_x/dy > 0 \quad \text{in} \quad 0 < y < 1, \\ \text{(ii)} & \tau_x(1) > 0.\end{array}\right\} \qquad (4.6)$$

Consider first the interior solution (3.5) for D. When y takes the value unity, for any x, the first term is of course a constant for fixed λ, and the second term is a minimum by condition (i) of (4.6) and the original assumption that $d\tau_x/dy$ is zero at the northern boundary. Therefore

$$D^2_{\min} = h^2 - 2\lambda(1-x)\,\tau_x(1)$$

for some value of x. It follows from condition (ii) that this value of x is zero. By an argument similar to that of case (a) above it may be verified that there exists no point in the boundary layer with depth less than this.

The result follows that, if the wind stress satisfies the conditions (4.6), the upper-layer depth first surfaces in the north-west corner of the basin when $\lambda = \lambda_c$, where λ_c is the smallest root of

$$h^2(\lambda) - 2\lambda\tau_x(1) = 0. \tag{4.7}$$

5. The separated current

In accordance with the remarks of the previous section we consider only those stress functions which satisfy the conditions (4.6). The type of circulation that we envisage is then as follows. Owing to the negative wind-stress curl the sense of the circulation will be clockwise. Apart from the region of surfacing, which we expect to be confined to the north-west corner of the basin, the fluid everywhere in the ocean interior will drift slowly southward. There is no reason to suppose that the presence of a surfacing lower layer to the north will prevent the normal formation of a boundary current along the southern part of the western coast. This current will flow northwards until it approaches the region of vanishing upper-layer depth. Up to this point the boundary-layer solutions of § 3 will remain valid.

The demand for continuity will then force the current to separate from the coast. Large depth gradients are expected normal to the surfacing line; thus, if the surfacing line is sufficiently well behaved (ideally one hopes that it will be inclined roughly towards the north-east), these gradients will provide the geostrophic balance necessary for maintaining an intense separated current and its integrity will be preserved. This current will flow on a course adjacent to the surfacing line and will be responsible for returning the fluid to the ocean interior. Indeed, a boundary layer of some kind is unavoidable in the region of surfacing, because the interior flow will not in general satisfy the boundary conditions at the surfacing line.

We proceed to derive the boundary-layer equations for the separated current. Introduce orthogonal co-ordinates (r, s) in the separated layer with r measured along the inward normal to the surfacing line and s measured intrinsically along this curve in the direction of the flow. The equation of the surfacing line, $x = X(y)$ say, is yet to be determined. If the curvature of the surfacing line is small the co-ordinate system (r, s) is, locally, approximately Cartesian. Let the velocity components in this frame be (U, V).

In order to illustrate the insensitivity of the position of the surfacing line to the internal dynamics of the boundary layer, the inertial terms are included in our discussion. On making the usual boundary-layer approximations to the momentum equations (2.10), one obtains, for the separated layer,

$$-fDV = -D\frac{\partial D}{\partial r}, \tag{5.1}$$

$$R_s D\left(U\frac{\partial V}{\partial r}+V\frac{\partial V}{\partial s}\right)+fDU = -D\frac{\partial D}{\partial s}-\epsilon V+\lambda\tau_s, \tag{5.2}$$

where τ_s is the component of τ_x in the direction of the local s axis. In these new co-ordinates the stream function is defined by

$$DU = -\partial\psi/\partial s, \quad DV = \partial\psi/\partial r, \tag{5.3}$$

and the boundary conditions are

$$D = 0 \quad \text{at} \quad r = 0, \tag{5.4}$$

$$D \to D_1(s), \quad \psi \to \psi_1(s) \quad \text{as} \quad r \to \infty. \tag{5.5}$$

The functions ψ_1 and D_1 are given by the interior solutions (3.4) and (3.5) at $x = X(y)$ and are not known explicitly until the function $X(y)$ is fixed.

The boundary condition (5.4) together with the definition of the stream function imply that ψ is constant at $r = 0$; that is, the surfacing line is a streamline. This constant may be equated to zero if the streamline is identified as that one which originates from the basin perimeter. (Strictly, this should be regarded as an assumption to be justified *a posteriori*.) Then the geostrophic relation (5.1) integrates to

$$\psi = D^2/2f. \tag{5.6}$$

On applying the matching conditions (5.5) and using the interior solutions (3.4) and (3.5), one obtains

$$\lambda(1-x)\frac{d\tau_x}{dy} = \frac{h^2}{2f}+\lambda(1-x)f\frac{d}{dy}\left(\frac{\tau_x}{f}\right), \quad \text{at} \quad x = X(y).$$

This gives the following equation for the line of surfacing:

$$x = X(y) = 1-\frac{h^2(\lambda)}{2\lambda\tau_x}. \tag{5.7}$$

We are now in a position to study the precise implications of the wind-stress conditions (4.6) that we have assumed. First, it is evident from the form of $X(y)$ above that a surfacing line can appear within the basin only in regions of positive τ_x. The conditions (4.6) assert that just one such region exists and that it is confined exclusively to the northern end of the basin. Secondly, since the wind-stress curl is negative, the gradient dy/dx of the surfacing line is positive, implying that the separated current is always inclined roughly towards the north-east. These two conclusions confirm our original expectation that the surfacing line separates the upper-layer fluid from the north-west corner of the basin.

The perimeter of the upper layer is now known to within the function $h(\lambda)$, which may be computed for any particular choice of wind stress from the conservation condition (2.16). Again, it is consistent to neglect contributions from the boundary layers when working with this condition. Thus our knowledge of the function h can be extended to values of the wind-stress amplitude greater than the critical value λ_c.

Computations have been performed for the special case $\tau_x = -\cos \pi y$ that was studied in § 4 in connexion with the North Atlantic circulation. The completed graph of h against λ is illustrated in figure 2. Some possible lines of surfacing are depicted in figure 3 for various values of the wind-stress amplitude.

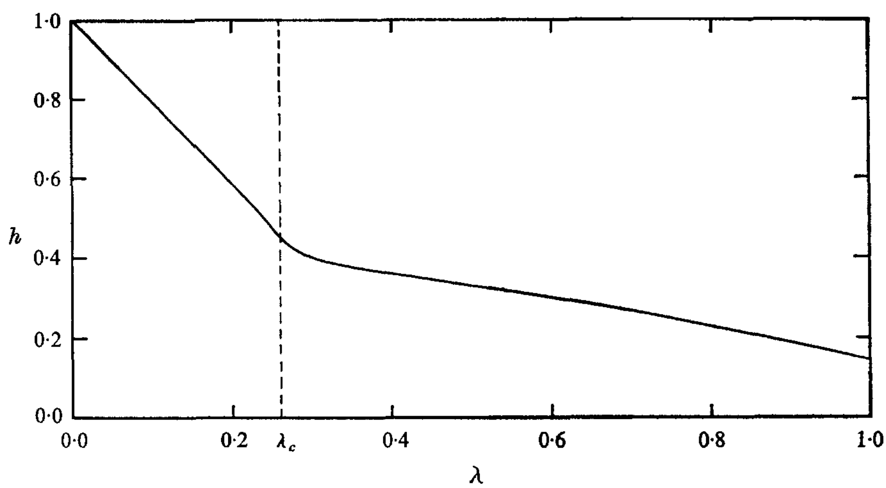

FIGURE 2. Layer depth at eastern coast against amplitude. Dependence extended beyond critical amplitude λ_c.

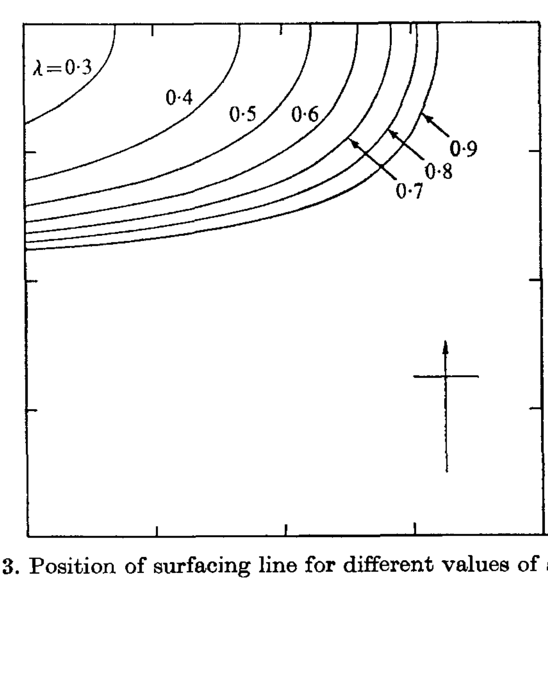

FIGURE 3. Position of surfacing line for different values of amplitude.

The functions $\psi_1(s)$ and $D_1(s)$ are now calculated to obtain

$$\psi_1 = \frac{h^2}{2\tau_x}\frac{d\tau_x}{dy}, \quad D_1^2 = \frac{h^2 f}{\tau_x}\frac{d\tau_x}{dy}. \tag{5.8}$$

The fact that all these results have been derived without reference to the second momentum equation stresses both the power of the geostrophic relation and the insensitivity of the surfacing position to the dynamics of the boundary layer. Moreover, since the functions ψ_1 and D_1 which appear in the boundary conditions (5.5) are known, the boundary-layer problem posed in the system (5.2) to (5.6) is completely isolated.

The fundamental equation for the separated boundary current is derived by eliminating the dependent variables in the momentum equation (5.2) in favour of the depth D. Then one obtains after some manipulation

$$R_s f J\left(\frac{D^2}{f}, \frac{1}{f}\frac{\partial D}{\partial r}\right) + 2\epsilon\frac{\partial D}{\partial r} + f'D^2 = f'D_1^2, \tag{5.9}$$

where f' is the derivative with respect to s of the Coriolis parameter and J is the two-dimensional Jacobian operator. Consistent with the assumption that the curvature of the surfacing line is small, the function f' is locally constant.

It is not possible to solve this equation in complete generality, so, instead, two special cases are considered.

(a) Purely frictional flow

Suppose that the inertial terms are small in comparison with the frictional terms. Then, by neglecting those terms which involve the Rossby number, (5.9) reduces to

$$2\epsilon\, \partial D/\partial r + f'D^2 = f'D_1^2. \tag{5.10}$$

In terms of a stretched transverse co-ordinate, $\eta = rf'D_1/\epsilon$, the solution satisfying $D = 0$ at $\eta = 0$ is

$$D = D_1(1-e^{-\eta})/(1+e^{-\eta}), \tag{5.11}$$

and the corresponding stream function is given by the geostrophic relation (5.6). Thus the width of the separated current is of order $\epsilon/f'D_1$, which is small everywhere except towards the northern boundary of the basin where D_1 tends to zero and the current breaks up.

This completes our analysis of the purely frictional circulation. The boundary-layer solutions for the separated current may be combined with the coastal and interior solutions of § 3 to obtain uniformly valid results.

As a numerical example the usual function $-\cos \pi y$ is chosen to describe the zonal wind stress. Some results are depicted in figure 4, which shows contour lines of the stream function. The numerical values $\epsilon = 0{\cdot}03$ and $\lambda/h^2 = 1$ were chosen as being illustrative, the latter corresponding to the values $\lambda = 0{\cdot}45$, $h = 0{\cdot}67$.

A comparison of figure 4 with charts of the observed North Atlantic circulation (see, for example, Stommel 1965) shows good qualitative agreement, particularly in the orientation of the separated current and in the gross features of the return flow. On referring to the definition of the dimensionless stress amplitude in (2.12) and assuming the typical values $L = 6 \times 10^8$ cm, $d = 3 \times 10^4$ cm, $g' = 2$ cm sec^{-2}, the above value of λ corresponds to a dimensional amplitude $W = 1{\cdot}35$. The discrepancy between this and the more realistic value of unity must be ascribed to the physical simplicity of the model.

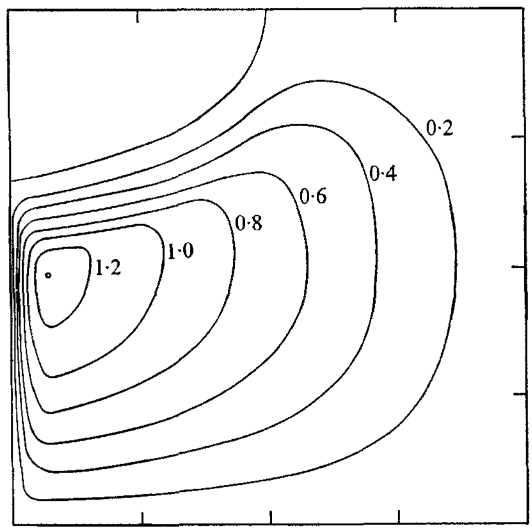

FIGURE 4. Contour lines of stream function for upper-layer transport when $\epsilon = 0{\cdot}03$, $\lambda = 0{\cdot}45$.

(*b*) *Purely inertial flow*

We now investigate the possibility of an inertially controlled separated current. Amongst the purely inertial solutions that (5.9) may possess, attention is confined to those which possess a similarity form. It may be expected that such solutions will be relevant sufficiently far away from the influence of initial conditions at the point of separation.

We write the dependent variable in the form

$$D = D_1(s)\,P(\eta), \tag{5.12}$$

where η is a scaled similarity variable defined by $\eta = rG(s)/\surd(2R_s)$ and P, G are functions to be determined. The boundary conditions in terms of the similarity function become

$$P(0) = 0, \quad P(\infty) = 1. \tag{5.13}$$

The following analysis demonstrates that no such solutions exist subject to these conditions.

If the similarity form is substituted into (5.9) with the frictional terms neglected, the equation for P takes the form

$$aP^2P'' - bPP'^2 - P^2 + 1 = 0, \tag{5.14}$$

where, for a similarity solution to exist, a and b must be constants which satisfy the relations

$$a = \frac{G^2}{\sqrt{f}}\frac{d}{dy}\left(\frac{D_1}{\sqrt{f}}\right), \quad b = G\frac{d}{dy}\left(\frac{D_1 G}{f}\right). \tag{5.15}$$

For the solution to exhibit the correct behaviour for large η we must have asymptotically

$$P(\eta) = 1 + p(\eta), \tag{5.16}$$

where p decays at least exponentially. After substituting this expansion into (5.15) and retaining only linear terms,

$$ap'' - 2p = 0, \tag{5.17}$$

from which it follows that a must be positive definite.

But, by the condition of geostrophy across the current and by the first of the relations (5.15), positive a implies that $d\psi_1/dy$ is also positive, which in turn implies that the separated current is gaining fluid from the ocean interior. This is in contradiction with the demand that the separated current be responsible for returning fluid to the ocean interior. One concludes that there exist no purely inertial similarity solutions satisfying the physical conditions of the problem.

It should be noted that this result for the separated current is in accord with the results of Greenspan (1962) and Pedlosky (1965) in their analyses of the inertial western boundary current attached to the coast. The correct interpretation of the result is that frictional effects cannot be neglected in the region of return flow.

6. General discussion of separated current

The analysis of the previous section demonstrates that the position of the separated boundary current can be calculated without actually considering its internal dynamics. This suggests that it should be possible to extract from the primitive equations important information about the region of surfacing without specifying precisely the mechanism by which the interior flow is closed.

Since we are interested primarily in the Gulf Stream problem we suppose that the wind stress is such that the separated current cuts across the north-west corner of the basin. Thus we avoid the more complicated situation of the totally internal region of surfacing, such as that predicted in case (*b*) of § 4. We know from the previous discussion that, for a stress which is purely zonal, independent of longitude, and which satisfies the sufficiency conditions (4.6), our requirement on the separated current is satisfied. Otherwise, the results will have to be tested for consistency *a posteriori*.

The formulation of the boundary-layer problems that we have discussed suffer from the basic objection that they do not accurately account for the physical processes of friction and inertia. The generality of the present discussion allows us to account for these effects in the momentum equations as precisely as we please, in terms of a single vector, $\mathbf{L}$ say. Then $\mathbf{L}$ may include the higher-order processes of lateral diffusion as well as those of internal friction and inertia. We assume that $\mathbf{L}$ is negligibly small everywhere except in certain narrow boundary-layer regions. For this assumption to be realistic it is necessary that the effects of inertia be not so great that they predominate over viscous effects (see Veronis (1966) and Stewart (1964)).

For the purposes of our discussion we divide the boundary layers into two classes. Those of the first class contain intense currents, of singularly high velocity, which we assume are confined to regions immediately adjacent to the western coast and surfacing line. Those of the second class are purely viscous layers necessary merely to satisfy the real fluid boundary condition of no-slip at other coastal boundaries.

At the same time, it may be argued that the forcing term $\lambda\boldsymbol{\tau}$ is not realistic within the separated current, for example in regions of vanishing upper-layer depth. Thus we introduce a generalized forcing term, $\lambda F\boldsymbol{\tau}$ say, with the single restriction that F is unity in all regions exterior to the separated current.

Note that none of these generalizations alters the dynamic balance in the ocean interior. The horizontal momentum equations for the upper-layer fluid may now be written as

$$\lambda F\boldsymbol{\tau} = \mathbf{f}\wedge D\mathbf{u} + D\nabla D + \mathbf{L}. \tag{6.1}$$

We take the curl of this equation and integrate over some surface S which is bounded by a closed curve C lying entirely within the upper layer. This results in the following statement of the vorticity balance inside C:

$$\lambda\iint \mathbf{n}\,.\,\mathrm{curl}\,(F\boldsymbol{\tau})\,dS = \oint(\mathbf{f}\wedge D\mathbf{u})\,.\,d\mathbf{l} + \oint \mathbf{L}\,.\,d\mathbf{l}, \tag{6.2}$$

where $\mathbf{n}$ is a unit vector in the direction of the upward vertical, and the line integrals are taken round C in a positive sense. The first and second terms represent, respectively, the total rate of input of wind-induced vorticity over the surface S and the rate of convection of planetary vorticity across the bounding curve C. The last term includes viscous diffusion and non-linear convection of relative vorticity across C, and possibly the effects of frictional dissipation. If S lies entirely within the ocean interior this last term does not contribute to the vorticity balance.

Now, suppose that the curve C enclosing the surface S is defined by the contour $ABCD$ (see figure 5), where AB and CD are lines of constant latitude, AD is adjacent to the surfacing line $x = X(y)$ and lies entirely within the separated current, and BC lies just outside the viscous layer at the eastern coast. Let A', B', C', D' be the corresponding intersections of the extended latitudinal lines AB, CD with the eastern coast and surfacing line. We consider the separate contributions of the two terms on the right-hand side of the vorticity equation (6.2). Although the interior flow cannot by itself satisfy the boundary condition of no slip at the eastern coast, it satisfies the condition of zero normal velocity. Thus we obtain

$$\oint(\mathbf{f}\wedge D\mathbf{u})\,.\,d\mathbf{l} = \left(\int_A^B + \int_B^C + \int_C^D + \int_D^A\right).\,(\mathbf{f}\wedge D\mathbf{u})\,.\,d\mathbf{l}$$

$$= [f\psi]_D^A + \int_D^A (\mathbf{f}\wedge D\mathbf{u})\,.\,d\mathbf{l}. \tag{6.3}$$

Also, it follows from the above remarks that the visco-inertial term contributes only along the line AD. Therefore, by employing the momentum equation (6.1),

$$\oint \mathbf{L}.d\mathbf{l} = \int_D^A (\lambda F\boldsymbol{\tau} - \mathbf{f}\wedge D\mathbf{u}).d\mathbf{l} - [\tfrac{1}{2}D^2]_D^A. \tag{6.4}$$

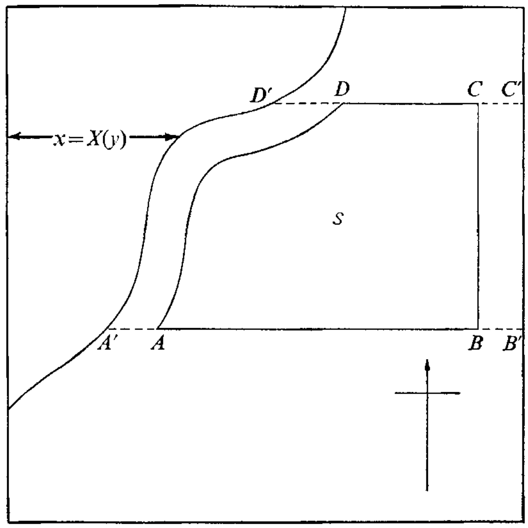

FIGURE 5. The contour C enclosing the surface S.

Substitution of these results into the vorticity equation obtains

$$\lambda\left(\int_A^B + \int_B^C + \int_C^D\right).\boldsymbol{\tau}.d\mathbf{l} = [f\psi - \tfrac{1}{2}D^2]_D^A. \tag{6.5}$$

Now let the points A, D tend to A', D' respectively. Then the right-hand side of (6.3) tends to $\psi_0[f]_D^A$, where ψ_0 is the constant value of the stream function at the surfacing line. (Note that this step is equivalent to assuming a geostrophic balance across the separated current.) By the original assumption that the streamline at the surfacing line originates from the basin perimeter, ψ_0 is zero. Therefore, since the replacement of B, C by B', C' in (6.5) incurs an error whose magnitude is no greater than the viscous boundary-layer thickness, we have

$$\left(\int_{A'}^{B'} + \int_{B'}^{C'} + \int_{C'}^{D'}\right)\boldsymbol{\tau}.d\mathbf{l} = 0. \tag{6.6}$$

Finally, this equation is true for any pair of latitudinal lines $A'B'$, $C'D'$ that make real intersections with the surfacing line, and is also independent of the mechanisms of viscosity and inertia. Thus, without loss of generality, (6.6) may be expressed as

$$\int_{X(y)}^1 \boldsymbol{\tau}_x\,dx + \int_y^1 [\tau_y]_{x=1}\,dy = k, \tag{6.7}$$

where, for a given wind stress, k is some function of amplitude only. This last identity formally determines the equation of the surfacing line to within the arbitrary function $k(\lambda)$, which can be fixed by a consideration of the interior flow.

For the special case $\boldsymbol{\tau} = (\tau_x(y), 0)$, (6.7) reduces to the form

$$X(y) = 1 - k(\lambda)/\tau_x, \tag{6.8}$$

which is in agreement with the result (5.7) obtained by matching techniques.

The important conclusion that we draw from the analysis of this section may be summarized as follows. If the effects of friction and inertia are confined to narrow regions of a boundary-layer character, then the surfacing line can be deduced from the condition of geostrophy and a knowledge of the interior flow alone. Thus, if ψ_I and D_I denote the solutions for the stream function and depth in the ocean interior, the position of the separated boundary current is formally determined by the identity $\psi_I = D_I^2/2f$, subject to appropriate conditions of consistency.

7. Conclusions

It has been shown that the two-layer model is capable of describing a separated boundary current. For a wind-stress function representative of conditions in the North Atlantic Ocean the theory predicts a circulation whose gross features are in good agreement with observation. Although the effects of friction and inertia are of critical importance within the separated current, they are of secondary importance in determining its position, the primary influence being the applied wind stress.

The present two-layer treatment lacks certain qualities of sophistication, but nevertheless includes the basic physical processes of the wind-driven ocean circulation, at the same time allowing a relatively simple analysis. The fundamental dependence of the position of the separated current on the applied wind stress suggests a degree of generality in the results, which should carry over to more complex models that include the non-linear effects of a continuous stratification.

The interesting case of the totally internal region of surfacing that is predicted for certain cyclonic wind systems requires further examination. It is intended to make this the subject of a future paper.

This work originally formed part of a doctoral thesis written at the University of Bristol while the author held a D.S.I.R. studentship.

REFERENCES

Bryan, K. 1963 A numerical investigation of a nonlinear model of a wind-driven ocean. *J. Atm. Sci.* **20**, 594–606.

Carrier, G. F. & Robinson, A. R. 1962 On the theory of the wind-driven ocean circulation. *J. Fluid Mech.* **12**, 49–80.

Greenspan, H. P. 1962 A criterion for the existence of inertial boundary layers in oceanic circulation. *Proc. Natn. Acad. Sci. U.S.A.* **48**, 2034–2039.

Moore, D. W. 1963 Rossby waves in ocean circulation. *Deep Sea Res.* **10**, 735–748.

Morgan, G. W. 1956 On the wind-driven ocean circulation. *Tellus*, **8**, 301–320.

Pedlosky, J. 1965 A necessary condition for the existence of an inertial boundary layer in a baroclinic ocean. *J. Mar. Res.* **23**, 69–72.

Stewart, R. W. 1964 The influence of friction on inertial models of oceanic circulation. *Studies in Oceanography* (Hidaka anniversary volume).

Stommel, H. M. 1948 The westward intensification of wind-driven ocean currents. *Trans. Am. Geophys. Union*, **29**, 202–206.

Stommel, H. M. 1965 *The Gulf Stream*. University of California Press.

Veronis, G. 1966 Wind-driven ocean circulation. Part 2. Numerical solutions of the nonlinear problem. *Deep Sea Res.* **13**, 31–55.

Welander, P. 1966 A two-layer frictional model of wind-driven motion in a rectangular oceanic basin. *Tellus*, **18**, 54–62.